Random resistor networks and Mott's variable range hopping

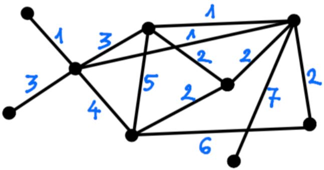
Alessandra Faggionato

University La Sapienza - Rome

Resistor network

Resistor network = weighted undirected (connected) graph

- $G = (\Lambda, E, c)$
- Λ = vertex set
- E = edge set
- $c : E \to (0, +\infty)$ conductivity function
  To simplify notation: $c_{x,y} = c(\{x, y\})$. Note: $c_{x,y} = c_{y,x}$



Associated random walk

Assumption: $\mu_x := \sum_{y:y\sim x} c_{x,y} < +\infty$ for all $x \in \Lambda$

RW associated to the resistor network

- $(X_t)_{t\geq 0}$: continuous-time random walk on Λ
- $c_{x,y}$: probability rate for a jump $x \curvearrowright y$
- The random walk waits at x an exponential time with mean $1/\mu_x$, afterwards it jumps to y with probability $p_{x,y} = \frac{c_{x,y}}{\mu_x}$
- $\big(\mu_x\big)_{x\in\Lambda}$: reversible measure

Potential theory (some glimpses)

• **Hitting times**

Hitting time of set $A \subset \Lambda$: $H_A := \inf\{t \geq 0 \,|\, X_t \in A\}$
Given disjoint $A, B \subset \Lambda$

$$\mathbb{P}_x(H_A < H_B) = V(x)$$

$V : \Lambda \to \mathbb{R}$: electrical potential with $V_{|A} \equiv 1$ and $V_{|B} \equiv 0$

• **Recurrence and transience**

Recurrence and transience of the RW $(X_t)_{t \geq 0}$ associated to an infinite resistor network have an electrical characterization

Some references

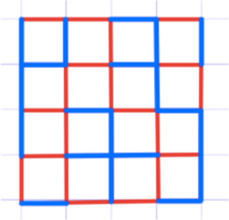
- P. Doyle, L. Snell; *Random Walks and Electric Networks*.
- A. Gaudilliere; *Condenser physics applied to Markov chains - A brief introduction to potential theory*.
- R. Lyons, Y. Peres; *Probability on Trees and Networks*.

Random conductance model

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space
- Resistor network $G = (\Lambda, E, c)$:

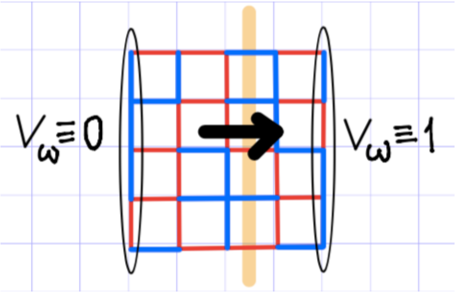
$$\Lambda := \mathbb{Z}^d \cap [-N, N]^d \,,$$
$$E := \big\{ \{x, y\} \in \Lambda \,:\, |x - y| = 1 \big\} \,,$$
$$c : \Omega \times E \to (0, +\infty) \,.$$

- Random conductivity function. We write $c_{x,y}(\omega)$



Effective conductivity

$\sigma(\omega)$:= electrical current flowing along the first direction when a unitary potential difference is applied to the left and right faces



V_ω: electrical potential
Current from x to $y \sim x$: $i_{x,y}(\omega) = c_{x,y}(\omega)\big(V_\omega(y) - V_\omega(x)\big)$

Effective conductivity

$V_\omega : \Lambda \to \mathbb{R}$ electrical potential. By Kirchhoff's law

$$\begin{cases} \sum_{y \in \Lambda : y \sim x} c_{x,y}(\omega)(V_\omega(y) - V_\omega(x)) = 0 & \text{if } x \in \Lambda,\ |x_1| \neq N\,, \\ V_\omega(x) = 0 & \text{if } x \in \Lambda,\ x_1 = -N\,, \\ V_\omega(x) = 1 & \text{if } x \in \Lambda,\ x_1 = N\,. \end{cases}$$

$$\sigma(\omega) = \sum_{x \in \Lambda_N : x_1 = -N} c_{x,x+e_1}(\omega)\big(V_\omega(x+e_1) - V_\omega(x)\big)$$

Difference operator

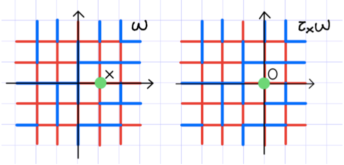
- Markov generator $\mathbb{L}_\omega f(x) := \sum_{y\in\Lambda: y\sim x} c_{x,y}(\omega)\big(f(y)-f(x)\big)$
- μ: counting measure on $\Lambda = [-N,N]^d \cap \mathbb{Z}^d$
- $\mathbb{L}_\omega : L^2(\Lambda,\mu) \to L^2(\Lambda,\mu)$ symmetric operator
- V_ω solves the elliptic equation with boundary conditions

$$\begin{cases} \mathbb{L}_\omega V_\omega(x) = 0 & \text{if } x\in\Lambda,\ |x_1|\neq N\,, \\ V_\omega(x) = 0 & \text{if } x\in\Lambda,\ x_1 = -N\,, \\ V_\omega(x) = 1 & \text{if } x\in\Lambda,\ x_1 = N\,. \end{cases}$$

- $\sigma(\omega) = \frac{1}{2}\langle V_\omega, -\mathbb{L}_\omega V_\omega\rangle_{L^2(\Lambda,\mu)}$

Assumptions on the conductivity function

- For simplicity:
  $\Omega = (0, +\infty)^{\mathbb{E}^d}$, $\mathbb{E}^d :=$ {unoriented edges of $\mathbb{Z}^d$}, $c_{x,y}(\omega) = \omega_{x,y}$.
- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space ($\mathcal{F}$=Borel σ–algebra)
- Action of $\mathbb{Z}^d$: given $x \in \mathbb{Z}^d$, $\tau_x \omega$ is the configuration ω translated by x



Assumptions

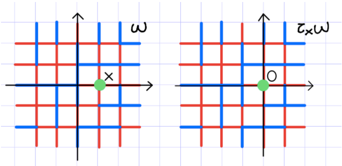
- $\mathbb{P}$ *is stationary and ergodic w.r.t. spatial translation*
- $\mathbb{E}[c_{x,y}] < +\infty$
- $e_1 = (1, 0, \dots, 0)$ *is an eigenvector of the effective homogenized matrix* D

Homogenized matrix D

D is the symmetric, positive semidefinite $d \times d$ matrix s.t. $\forall a \in \mathbb{R}^d$

$$a \cdot Da = \inf_{f \in L^\infty(\mathbb{P})} \frac{1}{2} \sum_{x:|x|=1} \int d\mathbb{P}(\omega) c_{0,x}(\omega) \left(a \cdot x - \nabla_x f(\omega)\right)^2$$

where $\nabla_x f(\omega) := f(\tau_x \omega) - f(\omega)$



Infinite volume limit

- Add N: $\Lambda \rightsquigarrow \Lambda_N$, $V_\omega \rightsquigarrow V_{N,\omega}$, $\sigma(\omega) \rightsquigarrow \sigma_N(\omega)$
- $\Lambda_N = [-N,N]^d \cap \mathbb{Z}^d$, $\quad \frac{1}{N}\Lambda_N \subset [-1,1]^d$
- $V^*_{N,\omega}$: the potential $V_{N,\omega}$ read on $\frac{1}{N}\Lambda_N$
- $\nu_N := N^{-d} \sum_{x \in \frac{1}{N}\Lambda_N} \delta_x$, $\quad \nu_N \to dx$ on $[-1,1]^d$

Let $\psi : [-1,1]^d \to [0,1]$, $\psi(x) := \frac{x_1+1}{2}$.

Theorem (AF, 2021+)

For $\mathbb{P}$–a.a. ω we have

$$\lim_{N \to +\infty} (2N)^{2-d} \sigma_N(\omega) = D_{1,1}\,.$$

Moreover, if $D_{1,1} > 0$, then for $\mathbb{P}$–a.a. ω

- $L^2(\nu_N) \ni V^*_{N,\omega} \to \psi \in L^2([-1,1]^d, dx)$,
- $\|V^*_{N,\omega} - \psi\|_{L^2(\nu_N)} \to 0$.

Homogenization

Let $D_{1,1} > 0$

Effective elliptic equation with b.c.

$$\nabla \cdot (D\nabla V)(x) = 0 \text{ if } x \in (-1,1)^d$$

$$\begin{cases} V(x) = 0 & \text{if } x_1 = -1\,, \\ V(x) = 1 & \text{if } x_1 = 1\,, \\ D\nabla V(x) \cdot \mathbf{n}(x) = 0 & \text{if } x \in \partial(-1,1)^d \cap \{x_1 \neq \pm 1\} \end{cases}$$

It must be $V(x) = \psi(x)$. Homogenization:

- $V^*_{N,\omega} \to \psi$,
- $(2N)^{2-d}\sigma_N(\omega) = \langle V^*_{N,\omega}, -\mathbb{L}^*_{N,\omega} V^*_{N,\omega}\rangle_{L^2(\nu_N)} \to \langle \psi, -\nabla \cdot (D\nabla\psi)\rangle_{L^2([-1,1]^d, dx)}$

Comments

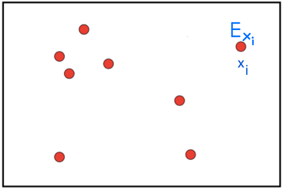
- D appears in other homogenization problems with different b.c.. See [AF20], Caputo & Ioffe AIHP 2003 (additional assumptions),...
- D =effective diffusion matrix for a weak CLT of the diffusively rescaled rw (X_t^ω) associated to the infinite resistor network. See [AF20]
- under additional assumptions, D = effective diffusion matrix for annealed/quenched invariance principle of (X_t^ω). See De Masi et al. JSP 1989, Biskup & Prescott EJP 2007, Mathieu JSP 2008, ...

[AF20]= A.F.; *Stochastic homogenization of random walks with a reversible measure in a random environment*. arXiv:2009.08258

Comments

- The proof of the theorem is an adaptation of A.F. *Miller–Abrahams random resistor network, Mott random walk and 2-scale homogenization.* arXiv:2002.03441
- The proof for $D_{1,1} > 0$ is based on stochastic homogenization by 2-scale convergence.

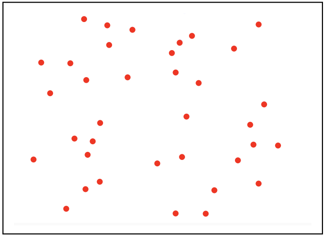
Marked simple point process



- simple point process $\{\bullet\} = \{x_i\}$: random locally finite subset of $\mathbb{R}^d$
- E_{x_i}: mark of x_i, random variable ($\{E_{x_i}\}$ are i.i.d.)
- $\omega = \{(x_i, E_{x_i})\}$ marked simple point process

Example: marked Poisson point process

- $\{x_i\}$: homogeneous Poisson point process on $\mathbb{R}^d$ with intensity m
  1. $A, B \subset \mathbb{R}^d$ and $A \cap B = \emptyset \Longrightarrow$
     $\big|\{x_i\} \cap A\big|$ and $\big|\{x_i\} \cap B\big|$ are independent
  2. $A \subset \mathbb{R}^d$ bounded $\Longrightarrow \big|\{x_i\} \cap A\big|$ is Poisson$(m\ell(A))$,
     $\ell(\cdot)$: Lebesgue measure



- We mark x_i by i.i.d. r.v.'s E_{x_i}

Miller–Abrahams (MA) resistor network

$\omega = \{(x_i, E_{x_i})\}$

Definition

The **MA random resistor network** is given by the complete graph on $\{x_i\} \subset \mathbb{R}^d$ with conductivity function

$$c_{x,y}(\omega) = \exp\Big\{ -|x-y| - \beta(|E_x| + |E_y| + |E_x - E_y|) \Big\}$$

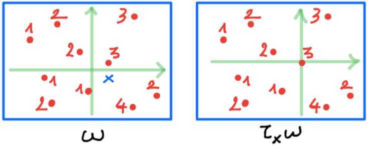
- **Mott random walk**: associated random walk $(X_t^\omega)_{t\geq 0}$. Hopping mechanism = Mott variable range hopping
- **Motivation**: electron transport in amorphous solids in the regime of strong Anderson localization

Effective homogenized matrix $D(\beta)$

$D(\beta)$: symmetric, positive semidefinite $d \times d$ matrix s.t. $\forall a \in \mathbb{R}^d$

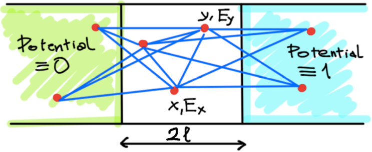
$$a \cdot D(\beta) a = \inf_{f \in L^\infty(\mathbb{P}_0)} \frac{1}{2} \int d\mathbb{P}_0(\omega) \sum_{x \in \{x_i\}} c_{0,x}(\omega) \left(a \cdot x - \nabla_x f(\omega)\right)^2$$

where $\omega = \{(x_i, E_{x_i})\}$, $\nabla_x f(\omega) := f(\tau_x \omega) - f(\omega)$



$\mathbb{P}_0$=Palm distribution associated to $\mathbb{P}$.
Roughly, $\mathbb{P}_0 = \mathbb{P}(\cdot | 0 \in \{x_i\})$
▷ $\forall f \in L^\infty(\mathbb{P}_0)$ we have an upper bound of $a \cdot D(\beta) a$

Homogenization



- At cost to change coordinate: D diagonal
- $\sigma_\ell(\omega)$= current along 1st direction in the box $(-\ell,\ell)^d$ under unitary electrical potential difference
- $m := \mathbb{E}\Big[\sharp(\{x_i\}\cap[0,1]^d)\Big]$

Theorem (F. arXiv:2002.03441 (2020))

For $\mathbb{P}$*–a.a.* ω, $\sigma_\infty(\beta) := \lim_{\ell\to+\infty}(2\ell)^{2-d}\sigma_\ell(\omega) = mD_{1,1}(\beta)$.

Again, homogenization for the electrical potential if $D_{1,1} > 0$.

Mott's law

E_i i.i.d. with common distribution $\nu_\alpha(dx) = c(\alpha)|x|^\alpha dx$ on $[-1,1]$, $\alpha \geq 0$
Physical law for $d \geq 2$ concerning low temperature decay of conductivity

$$\sigma_\infty(\beta) \asymp A(\beta) \exp\left\{ -c_0 \beta^{\frac{1+\alpha}{1+\alpha+d}} \right\}, \quad \beta \gg 1$$

- $\beta = 1/k_B T$: inverse temperature
- $A(\beta)$: weakly dependent from β
- $c_0 > 0$: β–independent physical parameter of the solid

Rigorous results for Mott's law $d \geq 2$

Mark distribution: $\nu_\alpha(dx) = c(\alpha)|x|^\alpha dx$ on $[-1, 1]$, $\alpha \geq 0$

- Content of [AF2021+]:
  - Rigorous proof of upper and lower bounds in agreement with Mott's law for a large class of marked point process for ν_α and $\nu_\alpha^+(dx) := c(\alpha)|x|^\alpha dx$ on $[0, 1]$, $\alpha \geq 0$
  - Rigorous proof of Mott's law for $\nu_\alpha^+(dx)$
  - Rigorous proof of Mott's law for $\nu_\alpha(dx)$ under a suitable percolation-type Ansatz
- Previous lower and upper bounds: (LB) A.F., D.Spehner, H.Schulz-Baldes CMP 2006, (UB) A.F., P.Mathieu CMP 2008
- For $d = 1$, different form of Mott's law. For LB and UB see A.F., Caputo AAP 2009
- All results are based on the variational characterization of $D(\beta)$

Main ingredients

- **Homogenization results**
- **Limit theorems** for rescaled thinned marked simple point processes towards marked Poisson point process
- **Percolation results** for the MA resistor network on marked Poisson point processes (in collaboration with H.A. Mimun)

Reason of thinning:

Recall that

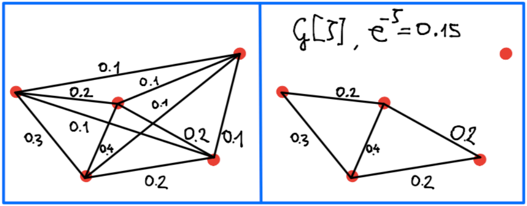
$$c_{x,y}(\omega) = \exp\Big\{ -|x-y| - \beta(|E_x| + |E_y| + |E_x - E_y|)\Big\}$$

As $\beta \to +\infty$, only nodes x with suitably small energy mark E_x contribute to electron transport

Critical path analysis

For simplicity: isotropic media

$\mathcal{G}[\zeta]$: resistor network with nodes $\{x_i\}$ end edges $\{x_i, x_j\}$ with $c_{x_i,x_j}(\omega) \geq e^{-\zeta}$



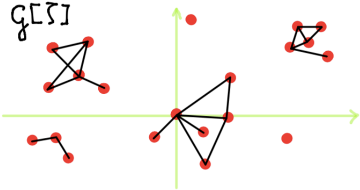
One expects: $\exists \zeta_c(\beta)$ such that, $\mathbb{P}$–a.s.,

$$\mathcal{G}[\zeta] \begin{cases} \text{percolates if } \zeta > \zeta_c(\beta) \\ \text{does not percolate if } \zeta < \zeta_c(\beta) \end{cases}$$

and

$$\boxed{\sigma_\infty(\beta) \asymp e^{-\zeta_c(\beta)}}$$

Subcritical phase



Theorem (A.F., H.A. Mimun, ALEA 2019)

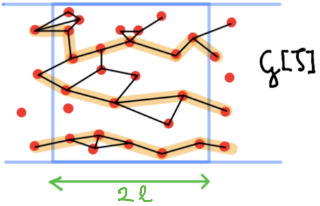
Let $\mathbb{P}$ be the law of a homogeneous Poisson point process marked by iid rv's. Let $\zeta < \zeta_c(\beta)$. Then $\exists c = c(\zeta) > 0$ such that $\forall n \in \mathbb{N}$

$$\mathbb{P}_0(\textit{diameter of cluster of } 0 \textit{ in } \mathcal{G}[\zeta] > n) \leq e^{-cn}.$$

- Function f in the variational formula of $D(\beta)$ for good upper bound
- Sharp phase transition for other random graphs on marked PPP
- Randomized algorithms, OSSS inequality (Duminil-Copin et al)

Supercritical phase

$\sigma_\infty(\beta) := \lim_{\ell\to+\infty}(2\ell)^{2-d}\sigma_\ell(\omega) = mD_{1,1}(\beta)$



Theorem (A.F., A.H. Mimun, arXiv:1912.07482)

Let $\mathbb{P}$ be the law of a homogeneous Poisson point process marked by nonnegative iid rv's. Let $\zeta > \zeta_c(\beta)$. Then $\exists c_1 = c_1(\zeta)$, $c_2 = c_2(\zeta) > 0$ such that $\forall \ell \in \mathbb{N}$

$$\mathbb{P}(\exists\, c_1\ell^{d-1} \textit{disjoint LR crossings of } \mathcal{G}[\zeta] \textit{ in } [-\ell,\ell]^d) \le 1 - e^{-c_2\ell^{d-1}}\,.$$

It allows to get LB on $\sigma_\ell(\omega)$

Supercritical phase

- The theorem holds for h–generalized Poisson Boolean models
- Based on a Renormalization procedure, which needs FKG inequality
- Next step (in progress): remove nonnegative mark assumption

Under the Ansatz that the above theorem holds for marks of general sign, Mott's law follows also for marks with law $\nu_\alpha(dx) = c(\alpha)|x|^\alpha dx$ on $[-1, 1]$, $\alpha \geq 0$ as shown in [AF, 2021+]

Wishing a lot of success to the UMI-group PRISMA,

THANKS !